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Generalized adiabatic invariants in classical mechanics

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Abstract. In this paper we present a method to obtain adiabatic invariants of any desired order in classical mechanics. The technique is quite similar to the one used in quantum mechanics, but the role of the 'spinning axis representation' is played now by a special set of canonical transformations.

1. Introduction

In quantum mechanics three adiabatic theorems are known which allow one to solve practically any problem related to this topic. The first one (see Messiah 1960) states that a system which is initially in an eigenstate of a Hamiltonian that has a slow time dependence will remain, at the end of the evolution, in the eigenstate of the instantaneous Hamiltonian that is deduced from the initial one by continuity. The second adiabatic theorem (Garrido and Sancho 1962) investigates the conditions under which the preceding statement is valid to mth order in the powers of $1/T$, where T is the long time interval during which the evolution of the system takes place. The third one, or generalized adiabatic theorem (Garrido 1964), is really a method to obtain adiabatic invariants of any desired order without imposing special conditions upon the derivatives of the Hamiltonian.

In classical mechanics Lenard (1959) proved adiabatic invariance to all orders for the classical one-dimensional non-linear oscillator. There exist also operational techniques (Garrido 1960, 1961) which permit the introduction of the interaction picture. It is possible (Garrido and Gascón 1962) to give general criteria to be satisfied by a slowly time-dependent Hamiltonian in order to possess adiabatic invariants of the mth order. Summarizing, we may therefore say that, in a certain sense, an extension of the two first quantum-mechanical adiabatic theorems, valid in classical mechanics, has been possible.

But the very important generalized adiabatic invariance or, in other words, a procedure to obtain adiabatic invariants to any desired order without imposing special conditions upon the Hamiltonian, has not been achieved in classical mechanics. That will be the principal aim of the present work. It will be observed that the procedure is quite similar to the one used in quantum mechanics.

The paper starts with the introduction of some well-known properties of the action and angle variables in order to fix the notation that we are going to use. We then find the generalized adiabatic invariants. Finally, the paper shows how the adiabatic invariance of mth order in classical mechanics (Garrido and Gascón 1962) can be deduced from the present generalized one by simply imposing the appropriate extra conditions upon the derivatives of the Hamiltonian.

In the present work we shall assume that the parameter T is such that $1/T \to 0$. It implies an adiabatic or slow change in the evolution of the system. We shall say that Q is an adiabatic invariant to the mth order, if it is possible to find a positive constant M such that for the change ΔQ of the quantity Q in the time interval T the inequality

$$|\Delta Q| < \left(\frac{1}{T}\right)^m M$$

holds (Lenard 1959).

2. A canonical transformation

For the sake of simplicity we restrict ourselves to study classical systems with only one degree of freedom whose motion is periodic. It is well known that this can occur in two different ways: libration (closed orbits in phase space) or rotation (periodic orbits in phase space). The degenerate systems will be excluded from the present treatment.

The action variable J is defined as

$$J = \oint p\, dq \tag{1}$$

where $\oint$ indicates integration over one complete period corresponding to the q. If q is cyclic, p is constant and

$$J = 2\pi p = \text{const.}$$

The generalized coordinate conjugate to J is known as the angle variable ω.

Let us consider a system whose Hamiltonian depends explicitly, though slowly, on time. We designate by $H(\tau)$ the value of the Hamiltonian at the instant of time $t \equiv \tau T$, where T is a large parameter. The rate of evolution of the system from the time origin time $t = 0$ until the instant $t = T$ depends only on the parameter τ, since the Hamiltonian is a function of the fictitious time τ and indirectly, through the definition of τ only, of the real time t.

The Hamilton–Jacobi function which generates the canonical transformation from q and p to ω_1 and J_1 is defined by

$$\begin{aligned} p &= \frac{\partial}{\partial q} F_1(q, \omega_1; \tau) \\ J_1 &= -\frac{\partial}{\partial \omega_1} F_1(q, \omega_1; \tau) \end{aligned} \tag{2}$$

where we use the suffix to prepare the notation for generalizing the procedure. The new Hamiltonian (ter Haar 1961) is

$$H_1(\omega_1, J_1; \tau) = H(J_1; \tau) + \frac{1}{T}\frac{\partial}{\partial \tau} F_1(q, \omega_1; \tau) \tag{3}$$

since H does not depend on ω_1, after the canonical transformation F_1 has been performed.

3. Generalized adiabatic invariants

Equation (3) can be interpreted as follows: $H(J_1;\ \tau)$ represents the unperturbed Hamiltonian and $(1/T)(\partial/\partial\tau)F_1$ a perturbation which is small because $1/T \to 0$. We call $H(J_1; \tau)$ 'unperturbed motion' because it can be easily solved, since ω_1 is a cyclic variable. The solution is trivial:

$$\begin{aligned} J_1(\tau) &= J_1(0) \\ \omega_1(\tau) &= \omega_1(0) + T\int_0^{\tau} \frac{\partial}{\partial J_1} H(J_1; \tau')\, d\tau' \end{aligned} \tag{4}$$

and the constants of motion of $H(J_1; \tau)$ represent adiabatic invariants of the first order, although the change can be evaluated by the perturbation term $(1/T)(\partial/\partial\tau)F_1$.

In order to generalize the foregoing procedure, we are going to treat $H_1(\omega_1,\ J_1;\ \tau)$ as we have treated $H(q, p;\ \tau)$. The reader may like to compare this point with the analogous one in the demonstration of the generalized adiabatic invariance in quantum mechanics (Garrido 1964).

We now define the action variable J_1 as

$$J_2(\tau) = \oint J_1\, d\omega_1 \tag{5}$$

where the integration is to be carried over a complete period of ω_1. In the same way we perform the change of variables from (ω_1, J_1) to (ω_2, J_2) by means of the canonical transformation $F_2(\omega_1, \omega_2; \tau)$, defined by

$$\begin{aligned} J_1 &= \frac{\partial}{\partial \omega_1} F_2(\omega_1, \omega_2; \tau) \\ J_2 &= -\frac{\partial}{\partial \omega_2} F_2(\omega_1, \omega_2; \tau) \end{aligned} \tag{6}$$

ω_2 being the angle variable which represents the generalized coordinate conjugate to J_2. The Hamiltonian in terms of ω_2 and J_2 is

$$H_2(\omega_2, J_2; \tau) = H_1(J_2; \tau) + \frac{1}{T}\frac{\partial}{\partial\tau} F_2(\omega_1, \omega_2; \tau) \tag{7}$$

where we have taken into account that, after the transformation F_2 has been performed, H_1 will not contain ω_2.

Because F_2 represents a canonical transformation, the equations of motion hold in the new variables with the canonical form. Thus we obtain

$$\frac{dJ_2}{d\tau} = \dot{J}_2 = T\frac{\partial H_2}{\partial \omega_2} = \frac{\partial}{\partial \omega_2}\left(\frac{\partial F_2}{\partial \tau}\right). \tag{8}$$

The same result (8) can be obtained by direct derivation in (6). From (4) and (7), by following perturbation theory in classical mechanics, one can easily deduce the relation

$$J_2(\tau) = J_1(0) + \frac{1}{T} G_1(\tau) \tag{9}$$

where $G_1(\tau)$ is a bounded function which does not play an important role in the present work (see, for instance, ter Haar 1961). Comparison between (8) and (9) allows us to write

$$\frac{\partial F_2}{\partial \tau} = \mathrm{O}\left(\frac{1}{T}\right). \tag{10}$$

The last expression means that the perturbation term in (7) is of the order of $1/T^2$. The solutions for the unperturbed motion are

$$\begin{aligned} J_2(\tau) &= J_2(0) \\ \omega_2(\tau) &= \omega_2(0) + T\int_0^\tau \frac{\partial}{\partial J_2} H_1(J_2; \tau')\, d\tau' \end{aligned} \tag{11}$$

because ω_2 is a cyclic variable. Then the constants of the motion represented by (11) are adiabatic invariants of second order.

If we perform successively m canonical transformations of type (2) and (6), the last one will be defined by

$$\begin{aligned} J_{m-1} &= \frac{\partial}{\partial \omega_m} F_m(\omega_{m-1}, \omega_m; \tau) \\ J_m &= -\frac{\partial}{\partial \omega_{m-1}} F_m(\omega_{m-1}, \omega_m; \tau) \end{aligned} \tag{12}$$

being

$$\begin{aligned} J_{m-1} &= \oint J_{m-2}\, d\omega_{m-2} \\ J_m &= \oint J_{m-1}\, d\omega_{m-1} \end{aligned} \tag{13}$$

where $\oint$ indicates integration over one complete period corresponding to the ω_{m-2} and ω_{m-1}, respectively. The final Hamiltonian will be

$$H_m(\omega_m, J_m; \tau) = H_{m-1}(J_m; \tau) + \frac{1}{T}\frac{\partial}{\partial \tau} F_m(\omega_{m-1}, \omega_m; \tau). \tag{14}$$

The solutions of the unperturbed motions are

$$\begin{aligned} J_m(\tau) &= J_m(0) \\ \omega_m(\tau) &= \omega_m(0) + T\int_0^\tau \frac{\partial}{\partial J_m} H_{m-1}(J_m; \tau')\, d\tau' \end{aligned} \tag{15}$$

because ω_m is a cyclic variable. As all the performed transformations are canonical, Hamilton equations of motion hold. Therefore we obtain

$$\frac{dJ_m}{d\tau} = \dot{J}_m = T\frac{\partial H_m}{\partial \omega_m} = \frac{\partial}{\partial \omega_m}\left(\frac{\partial F_m}{\partial \tau}\right). \tag{16}$$

The formula analogous to (9) will now be

$$J_m(\tau) = J_{m-1}(0) + \frac{1}{T^{m-1}} G_{m-1}(\tau) \tag{17}$$

because, in the same way that the change from H to H_1 is realized by means of a perturbation proportional to $1/T$ and from H_1 to H_2 by another perturbation proportional to $1/T^2$, the transformation of H_{m-2} (whose action variable is J_{m-1}) into H_{m-1} (whose action variable is J_m) implies a perturbation proportional to $1/T^{m-1}$. Hence, combining (16) and (17), we find

$$\frac{\partial}{\partial \tau} F_m(\omega_{m-1}, \omega_m; \tau) = O\left(\frac{1}{T^{m-1}}\right). \tag{18}$$

We arrive at the conclusion that the perturbation which appeared in (14) is of order $1/T^m$, and therefore the constants of the unperturbed motion (15) are generalized adiabatic invariants to mth order. These are expressed as functions of the last coordinates (ω_m, J_m). To obtain the same quantities in terms of (q, p), it is necessary to realize the corresponding inverse transformations; one can clarify that with the following scheme:

$$(qpH) \underset{F_1^{-1}}{\overset{F_1}{\rightleftarrows}} (\omega_1 J_1 H_1) \underset{F_2^{-1}}{\overset{F_2}{\rightleftarrows}} (\omega_2 J_2 H_2) \underset{F_3^{-1}}{\overset{F_3}{\rightleftarrows}} ::: \underset{F_{m-1}^{-1}}{\overset{F_{m-1}}{\rightleftarrows}} (\omega_{m-1} J_{m-1} H_{m-1}) \underset{F_m^{-1}}{\overset{F_m}{\rightleftarrows}} (\omega_m J_m H_m). \tag{19}$$

It is interesting to observe that, as in quantum mechanics, the generalized adiabatic invariants deduced above depend explicitly on time, because the F_i transformation functions depend explicitly on the fictitious time τ.

4. Comparison with the adiabatic invariance of *m*th order

In a preceding paper (Garrido and Gascón 1962) it was shown that all the constants of motion of the initial Hamiltonian are adiabatic invariants of order m of the slowly varying Hamiltonian, provided that its $m-1$ first time derivatives are zero at the beginning and at the end of the interval T, which is supposed to be very large. (The analogous adiabatic theorem to mth order exists in quantum mechanics (see, for example, Garrido and Sancho 1962).) The comparison between this result and the present generalized adiabatic invariance of mth order will be reduced to showing that one obtains the adiabatic invariants of mth order from the generalized ones of the same order, when we add to this second invariance the extra conditions that the $m-1$ first time derivatives of the Hamiltonian are zero initially and finally.

In order to do this, we recall that for periodic motions the action variable is an adiabatic invariant of order m, provided that the $m-1$ first time derivatives of the Hamiltonian are zero at the beginning and at the end of the large interval during which the evolution of the system takes place. This condition may be written as

$$J_1(\tau) = J_1(0) + O\left(\frac{1}{T^m}\right). \tag{20}$$

Furthermore, we shall suppose, as usual (see, for example, ter Haar 1961) that the system is periodic in ω_1 with a period of unity, ω_1 being no longer a strictly linear function of time. Therefore, combining (2) with (20) and taking into account the above-mentioned linearity of ω_1, we find

$$\frac{\partial}{\partial \tau} F_1(q, \omega_1; \tau) = O\left(\frac{1}{T^{m-1}}\right). \tag{21}$$

The last expression indicates that the perturbation in (3) is now of the order of $1/T^m$. Hence the constants of motion represented by (4) are direct adiabatic invariants of mth order.

In the same way, by applying perturbation theory, one obtains from (3) and (21)

$$J_2(\tau) = J_1(0) + O\left(\frac{1}{T^m}\right). \tag{22}$$

Equations (6) and (22) yield the expression

$$\frac{\partial}{\partial\tau} F_2(\omega_1, \omega_2; \tau) = O\left(\frac{1}{T^{m-1}}\right). \tag{23}$$

From (7) and (23) we deduce that the constants of unperturbed motion in (7), whose solutions are represented by (11), are adiabatic invariants of mth order. This process can be repeated with the remaining Hamiltonians H_3, H_4, ..., H_{m-1}, in the same way as with H_1 and H_2. The degree of approximate validity of their respective adiabatic invariants cannot be improved: they are all adiabatic invariants of mth order. We have thus shown that the adiabatic invariance of mth order (which requires extra conditions imposed upon the Hamiltonian) can be obtained from the present generalized adiabatic invariance (which does not fulfil any special condition) by adding to this second adiabatic invariance the appropriate extra conditions mentioned above.

5. Conclusion

The present paper provides a general method for obtaining adiabatic invariants of any desired order in classical mechanics. Its application does not require special conditions to be imposed on the Hamiltonian. These invariants depend explicitly on time. When the $m-1$ first time derivatives of the Hamiltonian are zero initially and finally, the method enables us to deduce the well-known adiabatic invariance to mth order. In this case the quantities which are adiabatic invariants do not depend on time. To determine such quantities it is not necessary to iterate the procedure. Only the first canonical transformation must be performed, because the application of F_2, F_3, ..., F_m does not improve the degree of approximation of the adiabatic invariance, as we have shown.

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